Searching in a Graph

CS 5010 Program Design Paradigms
"Bootcamp"

Lesson 8.4

Introduction

- Many problems in computer science involve directed graphs.
- General recursion is an essential tool for computing on graphs.
- In this lesson we will design a program for an important problem on graphs, using general recursion
- The algorithm we will develop has many other applications.

Learning Objectives

- At the end of this lesson you should be able to:
  - explain what a directed graph is, and what it means for one node to be reachable from another
  - explain what a closure problem is
  - explain the worklist algorithm
  - write similar programs for searching in graphs.

What's a graph?

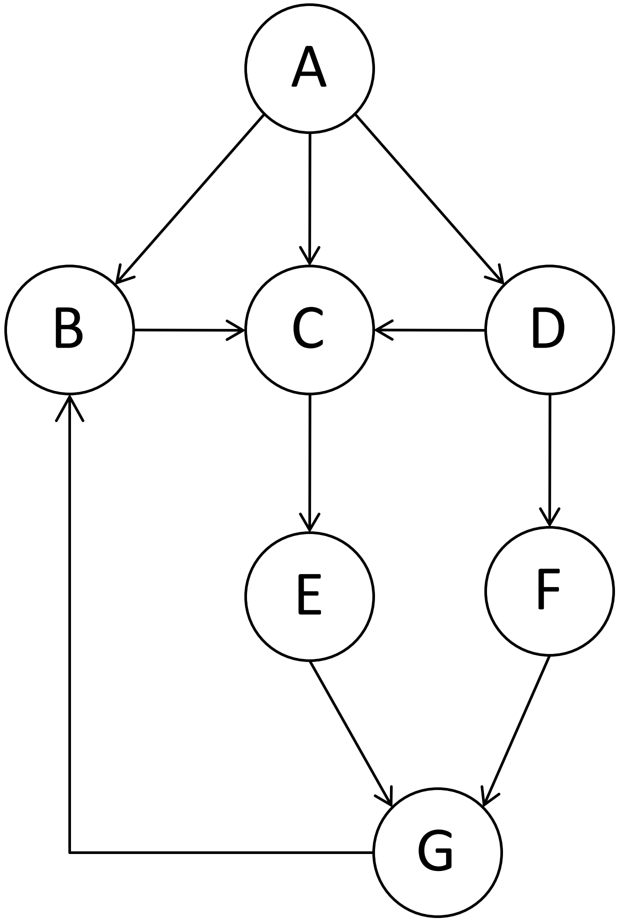
- You should be familiar with the notion of a graph from your previous courses.
- A graph consists of some nodes and some edges.
- We will be dealing with directed graphs, in which each edge has a direction. We will indicate the direction with an arrow.

A Graph

nodes: A, B, C, etc.

edges:

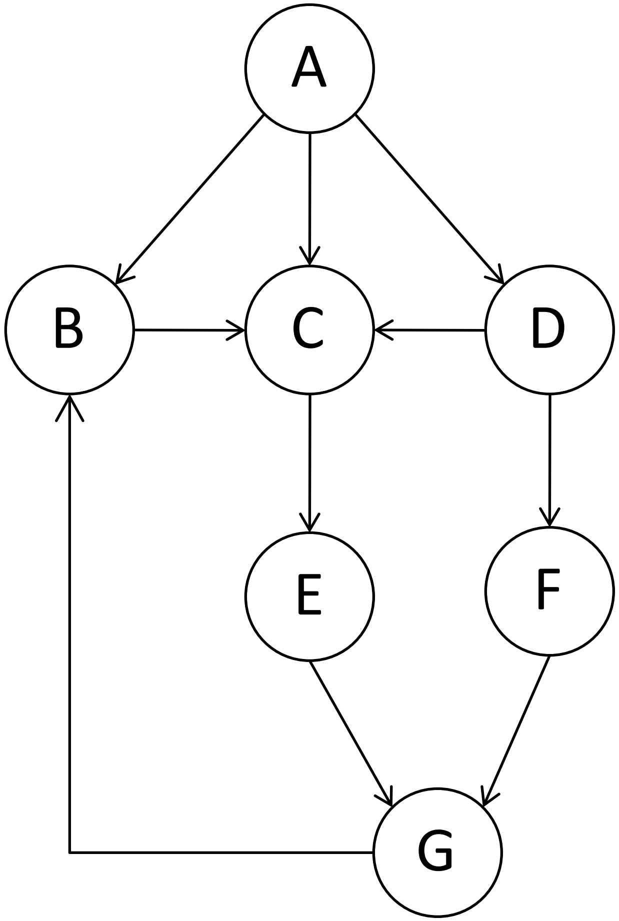
(A,B), (A,C),(A,D), etc.



successors of a node

The successors of a node are the nodes that it can get to by following one edge.

(successors A) = {B,C,D}
(successors D) = {C,F}

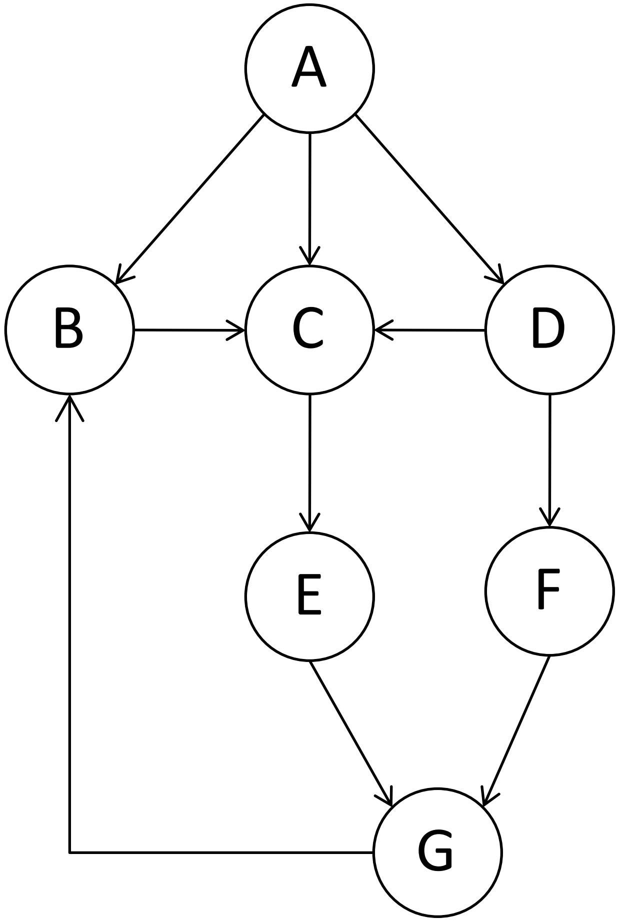


all-successors of a set of nodes

all-successors of a set of nodes are all the successors of any of the nodes in the set

(all-successors {}) = {}

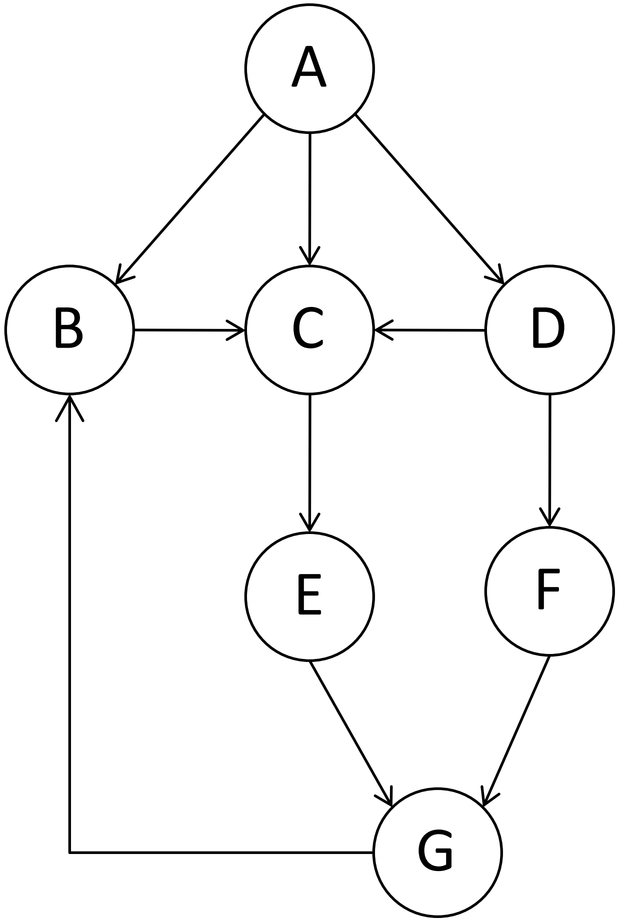
(all-successors {A,D})
= {B,C,D,F}



Paths in a Graph

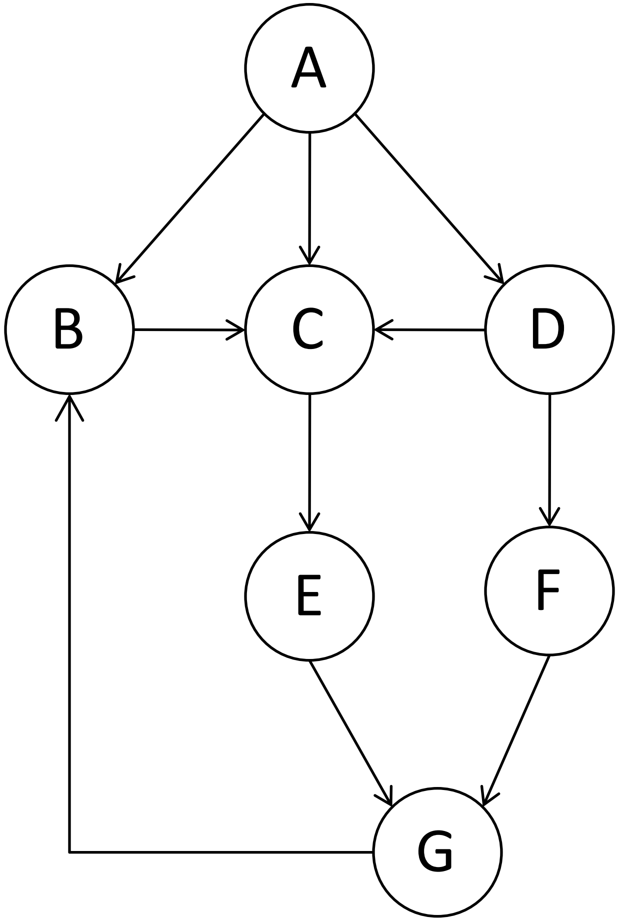
A path is a sequence of nodes that are connected by edges. Notice that the node A by itself is a path, since there are no edges to check. On the other hand, (A,A) is not a path, since there is no edge from A to itself.

paths:	non-paths:
(A,C,E)	(D, A)
(B,C,E,G)	(A,C,G)
(A,D,C,E)	(A,C,D,E)
(A)	(A,A)



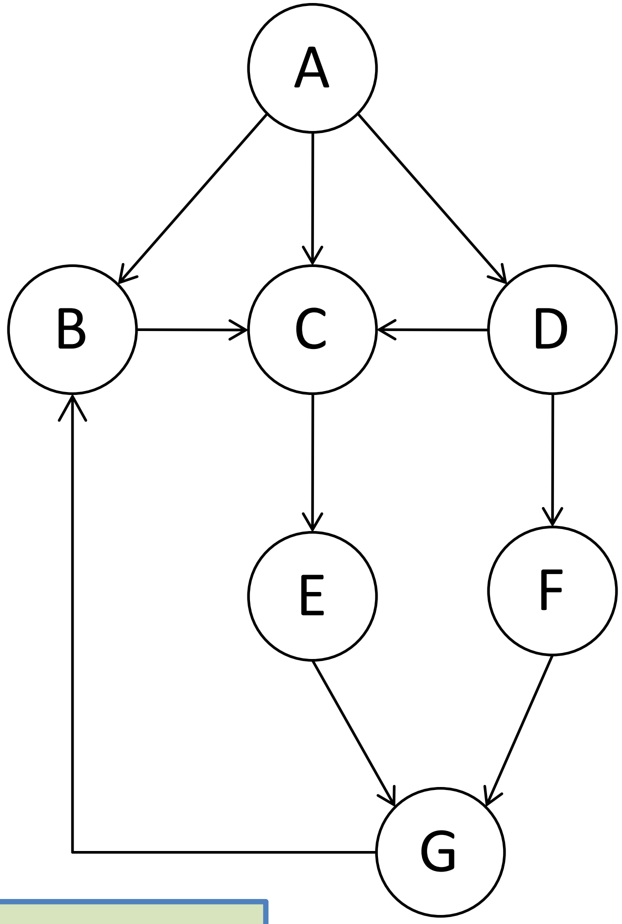
Cycles

This graph has a *cycle*: a path from the node B to itself. Graphs without cycles are said to be *acyclic*.
For this lesson, our graphs are allowed to have cycles.



Reachability

One node is *reachable* from another if there is a path from the one node to the other.



Nodes reachable from D:
{B,C,D,E,F,G}
Not reachable:
{A}

D is reachable from itself by a path of length 0, but not by any other path

Another classic application of general recursion

```
reachables :
  Graph SetOfNode -> SetOfNode
GIVEN: a graph and a set of nodes
RETURNS: the set of nodes that is
reachable from the given set of
nodes
```

Definition

A node t is reachable from a node s iff either

1. t = s
2. there is some node s' such that
    a. s' is a successor of s
    b. t is reachable from s'

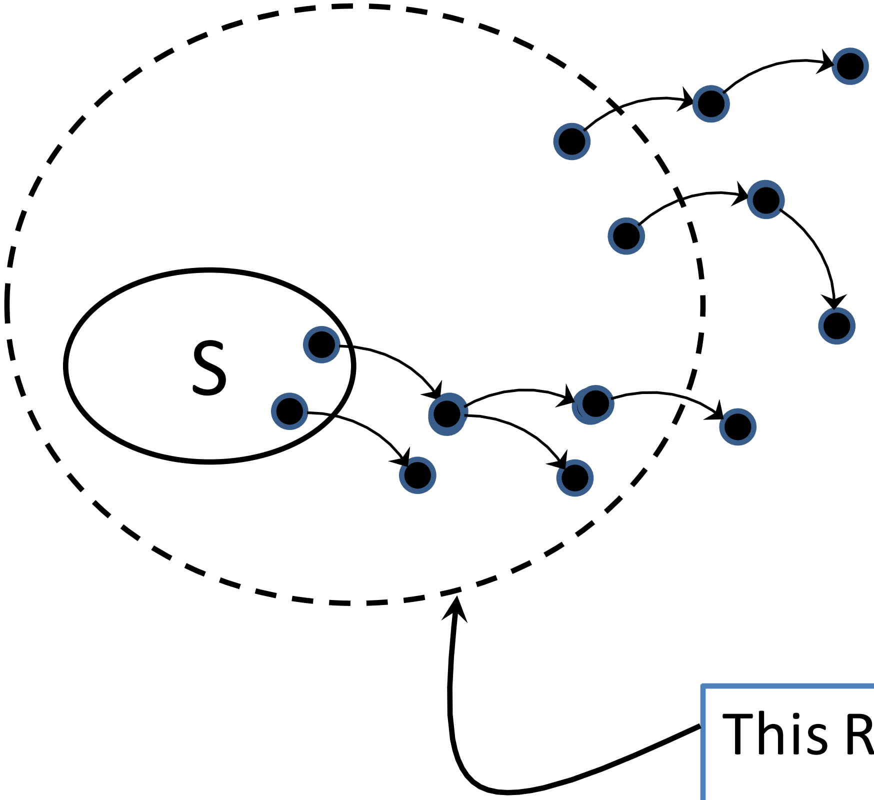
What does this definition tell us?

- If S is a set of nodes, then (reachables S) has the property that:
  - IF node n is in (all-successors (reachables S))
  - THEN n is already in (reachables S).
- Why? Because if n is a successor of a node reachable from S, then n is itself reachable from S

Another way of looking at this:

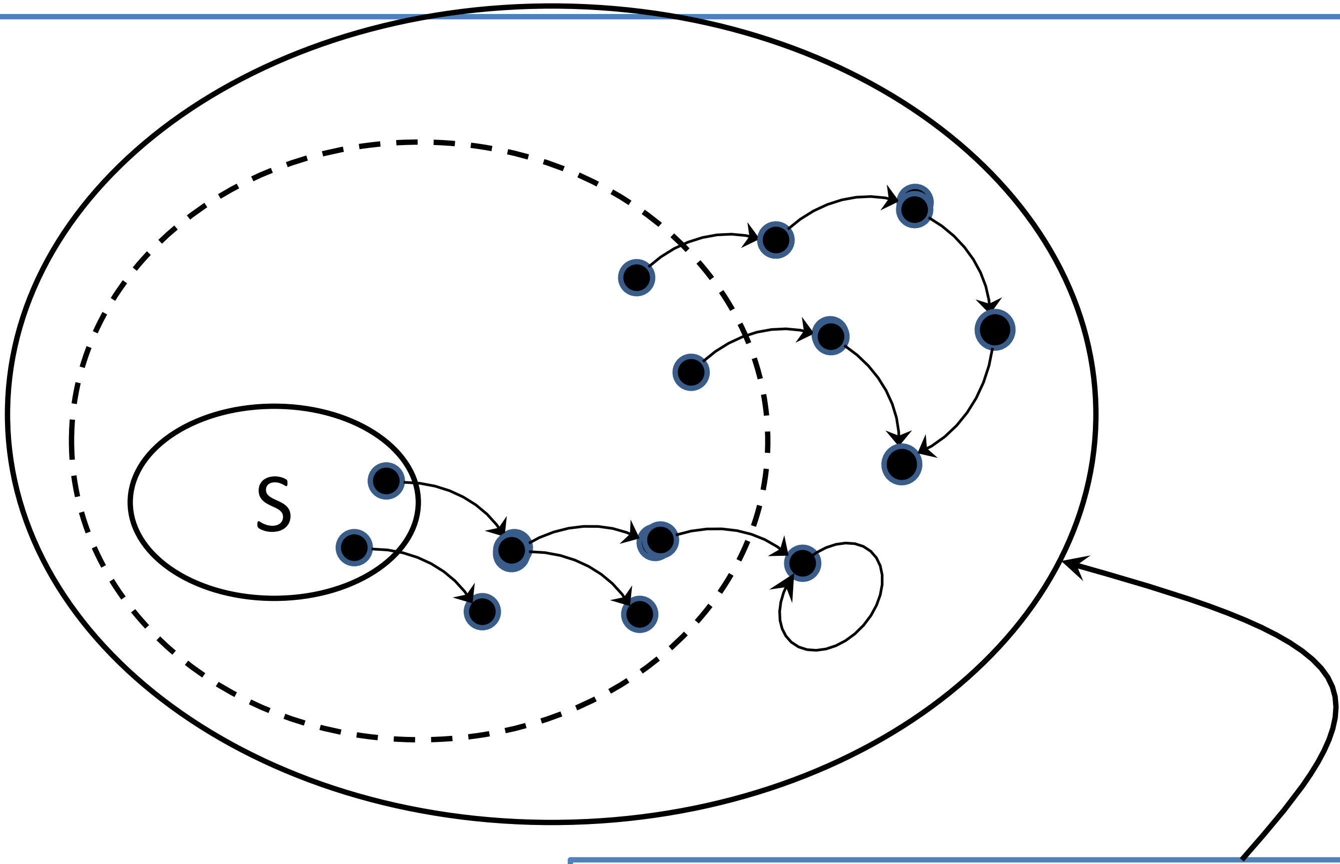
- If S is a set of nodes, then
  - (reachables S) is the smallest set R of nodes such that
  1. S is a subset of R
  2. (all-successors R) is a subset of R.

Growing (reachables S): not done yet



This R is not closed under successors: more reachables to be found!

Growing (reachables S): done!



The R contains S as a subset and is closed under successors. So it is (reachables S)

Closure problems

- This is called a "closure problem": we want to find the smallest set R which contains our starting set S and which is closed under some operation
- In this case, we want to find the smallest set that contains our starting set of nodes, and which is closed under **`all-successors`**.

Assumptions

- We assume we've got data definitions for Node and Graph, and functions
  - `node=? : Node Node -> Boolean`
  - `successors :`
    ```
    Node Graph -> SetOfNode
    ```
  - `all-successors :`
    ```
    SetOfNode Graph -> SetOfNode
    ```
- We also assume that our graph is finite.

Initial Solution

```
;; reachables: SetOfNode Graph -> SetOfNode
;; GIVEN: A set of nodes in a graph
;; RETURNS: the set of nodes reachable from the starting nodes
;; STRATEGY: recur on (nodes U their immediate successors)
;; HALTING MEASURE:
;;   # of nodes in the graph that are NOT in the set 'nodes'.
(define (reachables nodes graph)
  (local
    ((define candidates (all-successors nodes graph)))
    (cond
      [(subset? candidates nodes) nodes]
      [else (reachables
               (set-union candidates nodes)
               graph)])))
```

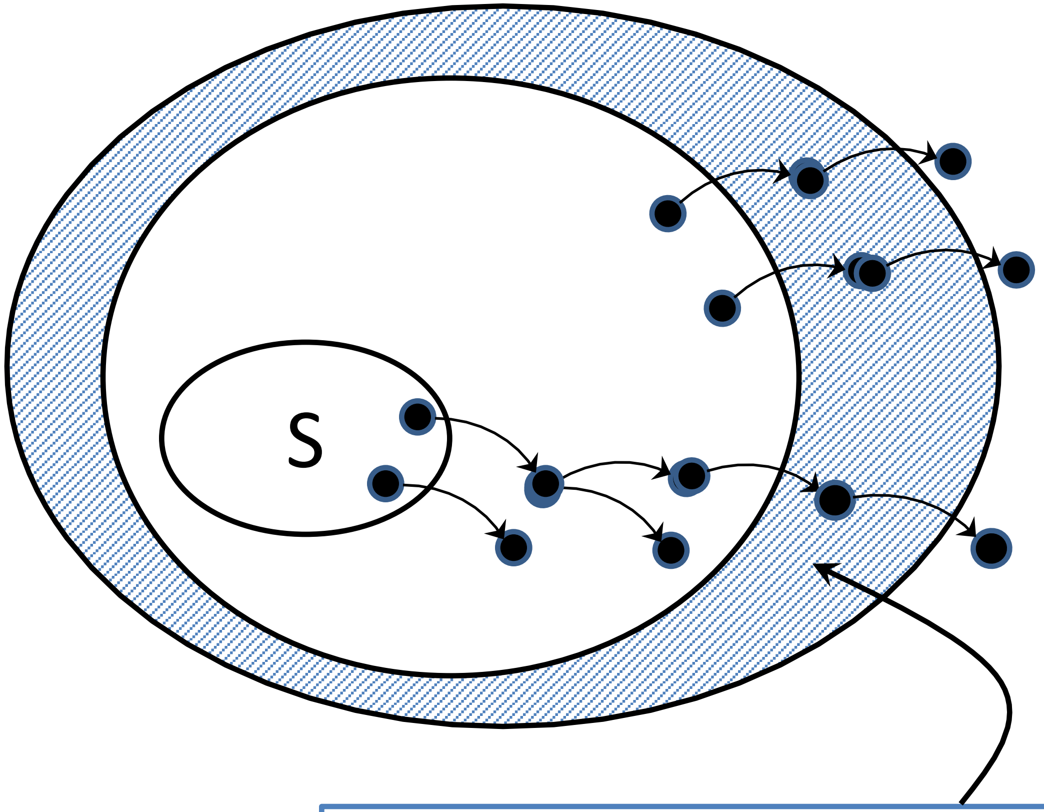
if 'nodes' is closed under all-successors, then we're done

Otherwise, add the candidates to the nodes, and try again

Problem with this algorithm

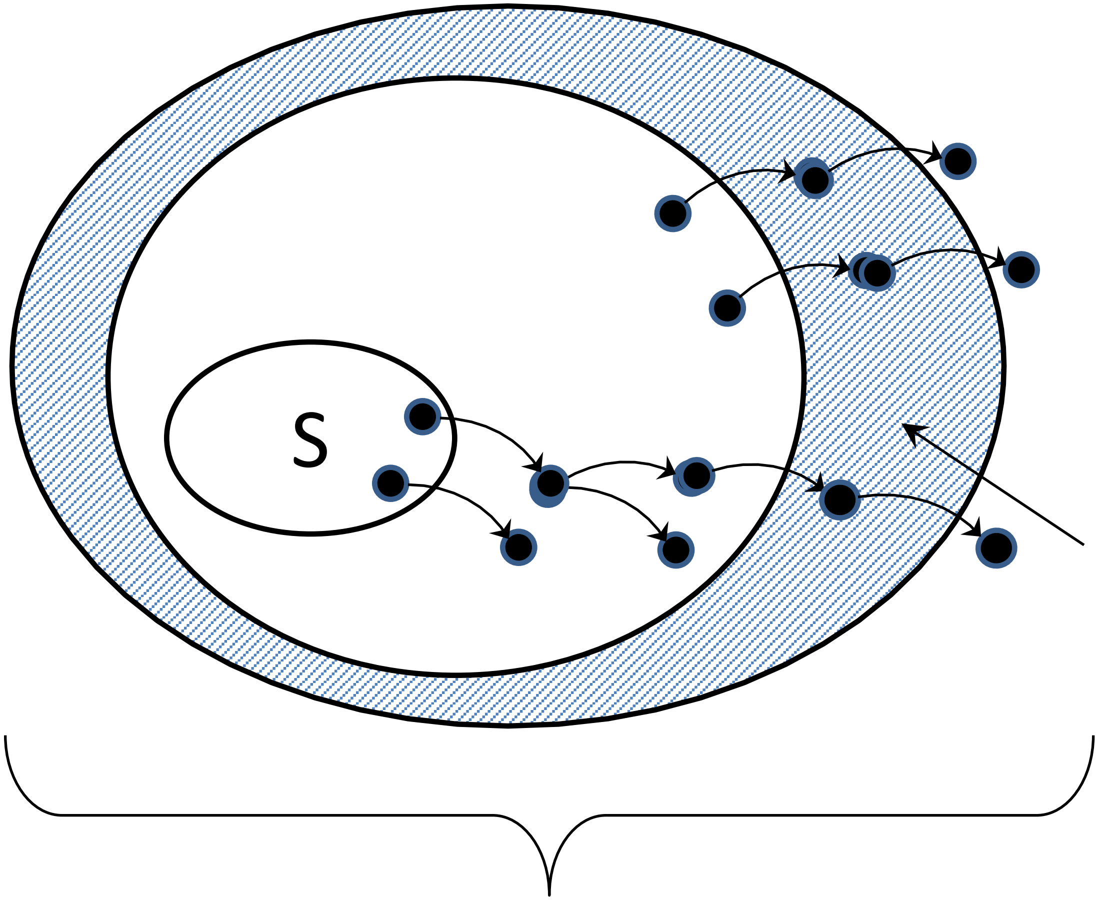
- We keep looking at the same nodes over and over again:
  - we always say **`(all-successors nodes)`**, but we've seen most of those nodes before.

A Better Idea: keep track of which nodes are new



only need to explore nodes in this region– all others are accounted for.

Do this with an extra argument and an invariant



newest = the most recently added elements of 'nodes'

nodes

Version with invariant

```
;; reachables1 : SetOfNode SetOfNode Graph
;; GIVEN: two sets of nodes, 'nodes' and 'newest' in a graph
;; WHERE: newest is a subset of nodes
;; AND: newest is the most recently added set of nodes
;; RETURNS: the set of nodes reachable from 'nodes'.
;; STRATEGY: recur on successors of newest that are not already in nodes;
;;    halt when no more successors
;; HALTING MEASURE:
;;   # of nodes in the graph that are NOT in the set 'nodes'.

(define (reachables1 nodes newest graph)
  (local
    ((define candidates (set-diff
                          (all-successors newest graph)
                          nodes)))
    (cond
      [(empty? candidates) nodes]
      [else (reachables1
              (append candidates nodes)
              candidates
              graph)])))
```

Since candidates is disjoint from nodes, we can replace the set-union with append.

Initializing the invariant

```
;; we initialize newest to nodes since
;; initially all the nodes are new.

;; STRATEGY: Call more general function

(define (reachables nodes graph)
  (reachables1 nodes nodes graph))
```

This is called the "worklist" algorithm

- It is used in many applications
  - in compiler analysis
  - in AI (theorem proving, etc.)

You could use this to define **path?**

```
;; path? : Graph Node Node -> Boolean
;; GIVEN: a graph and a source and a
;; target node in the graph
;; RETURNS: true iff there is a path in g
;; from src to tgt
;; STRATEGY: call more general function
(define (path? graph src tgt)
  (member tgt (reachables (list src) graph)))
```

But for that, you don't need to build the whole set

```
(define (path? graph src tgt)
  (local
    ((define (reachable-from? newest nodes)
       ;; RETURNS: true iff there is a path from src to tgt in graph
       ;; INVARIANT: newest is a subset of nodes
       ;; AND:
       ;;   (there is a path from src to tgt in graph)
       ;;   iff (there is a path from some node in newest to tgt)
       ;; STRATEGY:  recur on successors of newest; halt when tgt is found.
       ;; HALTING MEASURE: the number of graph nodes _not_ in 'nodes'
       (cond
         [(member tgt newest) true]
         [else (local
                 ((define candidates (set-diff
                                      (all-successors newest graph)
                                      nodes)))
                 (cond
                   [(empty? candidates) false]
                   [else (reachable-from?
                          candidates
                          (append candidates nodes))]))])))
    (reachable-from? (list src) (list src))))
```

Look carefully at this invariant.

Just watch for **tgt** to show up in newest

Why is the invariant true again at this call?

Can you check to see that the invariant is true at this call?

Another topic: changing the data representation

```
;; reachables: SetOfNode Graph -> SetOfNode
(define (reachables nodes graph)
  (local
    ((define candidates (all-successors nodes graph)))
    (cond
      [(subset? candidates nodes) nodes]
      [else (reachables
               (set-union candidates nodes)
               graph)])))
```

Notice that the only thing we do with graph is to pass it to all-successors.

So let’s pass in the graph’s all-successors function

```
;; reachables: SetOfNode (SetOfNode -> SetOfNode)
;;             -> SetOfNode
(define (reachables nodes all-successors-fn)
  (local
    ((define candidates (all-successors-fn nodes)))
    (cond
      [(subset? candidates nodes) nodes]
      [else (reachables
               (set-union candidates nodes)
               all-successors-fn)])))
```

How do you build an **all-successors-fn**?

```
;; You could do it from a data structure:

;; Graph -> (SetOfNode -> SetOfNode)
(define (make-all-successors-fn g)
  (lambda (nodes)
    (all-successors nodes g)))
```

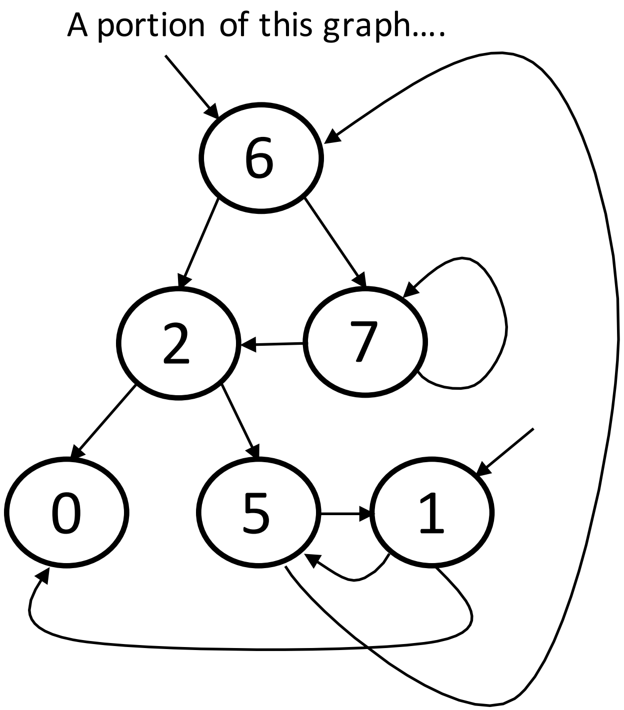
Or you could avoid building the data structure entirely

- Just define a successors function from scratch, and then define all-successors using a HOF.
- Good thing to do if your graph is very large– e.g. Rubik’s cube.

Example of an "implicit graph"

```
;; Int -> SetOfInt
;; GIVEN: an integer
;; RETURNS: the list of its successors in the implicit graph.
;; For this graph, this is always a set (no repetitions)
(define (successors1 n)
  (if (<= n 0)
      empty
      (local
        ((define n1 (quotient n 3)))
        (list n1 (+ n1 5)))))
```

From Examples/08-5a-implicit-graphs.rkt



```
;; all-successors1 : SetOfInt -> SetOfInt
;; GIVEN: A set of nodes
;; RETURNS: the set of all their
successors in our implicit graph
;; STRATEGY: Use HOFs map, then unionall.
(define (all-successors1 ns)
  (unionall (map successors1 ns)))
```

Here’s a function you could pass to **reachables**.

Summary

- We've applied General Recursion to an important problem: graph reachability
- We considered the functions we needed to write on graphs in order to choose our representation(s).
- We used list abstractions to make our program easier to write

Learning Objectives

- You should now be able to:
  - explain what a directed graph is, and what it means for one node to be reachable from another
  - explain how the function for reachability works.
  - explain what a closure problem is
  - explain the worklist algorithm
  - write similar programs for searching in graphs.

Next Steps

- Study 08-5-reachability.rkt and 08-5a-implicit-graphs.rkt in the Examples folder.
- If you have questions about this lesson, ask them on the Discussion Board
- Do Guided Practice 8.4